(Area)

Let R be the region bounded by the curves

$y = 0.5x(x+4)$ and $y = -x(x+4)$

Answer: 16

a) Sketch the region

b) Find the area of the region.

Volume.

Answer ≈ 85.734

Let R be the region bounded by the graph $y = e^{-x+2}$, the x-axis, the y-axis and the vertical line $x=4$, rotate the region about the x-axis.

(a) Sketch

(b) Find the volume of the solid of revolution

Average Values

What is the average value of the function $f(x) = 2x\sin(3x)$ on $[0, \pi]$

$\frac{2}{3}$

Definite Integrals

Find: (a) $\int_0^1 6t(3t^2-1)^5\,dt = 10.5$

(b) $\int_1^e x^3 \ln(x)\,dx = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}$

Case 3 | $\mathcal{D} < 0$, i.e $Q(x)$ has no (real) roots

Method: Completing the square in $Q(x)$.

$$\frac{P(x)}{Q(x)} = \frac{1}{x^2+2x+2} \quad ; \quad \int \frac{dx}{x^2+2x+2} - ?$$

$$x^2+2x+2 = 0$$

$$\mathcal{D} = 4 - 4\cdot 2 = -4 < 0.$$

$$\frac{1}{x^2+2x+2} = \frac{1}{(x^2+2x+1)+1} = \frac{1}{(x+1)^2+1}$$

$$\int \frac{dx}{(x+1)^2+1} = \boxed{\begin{array}{l}\text{we use the} \\ \text{substitution } u(x) = x+1 \\ \qquad du = dx\end{array}} =$$

$$= \int \frac{du}{u^2+1} = \arctan(u) + C = \arctan(x+1) + C.$$

$\mathcal{D} < 0.$

$\boxed{\text{Example 1}}$ $$\int \frac{3x+2}{x^2-2x+5}\,dx = \int \frac{3x+2}{(x^2-2x+1)+4}\,dx =$$

$$= \int \frac{3x+2}{(x-1)^2+4}\,dx = \boxed{\begin{array}{l}\text{the substitution } u(x) = x-1 \\ \qquad du = dx \\ \qquad x = u+1\end{array}}$$

$$= \int \frac{3(u+1)+2}{u^2+4}\,du = \int \frac{3u+5}{u^2+4}\,du =$$

$$= \underbrace{3\int \frac{u\,du}{u^2+4}}_{I_1} + \underbrace{5\int \frac{du}{u^2+4}}_{I_2} \quad ☆$$

$$I_1 = 3\int \frac{u\,du}{u^2+4} \quad \boxed{\begin{array}{l} u^2+4 = z(u) \\ 2u\,du = dz \quad \text{or} \quad u\,du = \frac{dz}{2} \end{array}}$$

$$= \frac{3}{2}\int \frac{dz}{z} = \frac{3}{2}\ln|z| + C = \frac{3}{2}\ln|u^2+4| + C.$$

$$(\arctan u)' = \frac{1}{u^2+1}$$

$$I_2 = 5\int \frac{du}{u^2+4} = 5\int \frac{du}{4\left(\frac{u^2}{4}+1\right)} = \frac{5}{4}\int \frac{du}{\frac{u^2}{4}+1} =$$

$$= \frac{5}{4}\int \frac{du}{\left(\frac{u}{2}\right)^2+1} = \boxed{\begin{array}{l} \frac{u}{2} = z \\ du = 2dz \end{array}} = \frac{5}{4}\int \frac{2dz}{z^2+1} =$$

$$= \frac{5}{2}\int \frac{dz}{z^2+1} = \frac{5}{2}\operatorname{artan}(z) + C = \frac{5}{2}\operatorname{artan}\left(\frac{u}{2}\right) + C.$$

$$\overbrace{\frac{3}{2}\ln|u^2+4|}^{I_1} + \overbrace{\frac{5}{2}\arctan\left(\frac{u}{2}\right)}^{I_2} + C =$$

$$= \frac{3}{2}\ln\left|(x-1)^2+4\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C =$$

$$= \frac{3}{2}\ln|x^2-2x+5| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

If $\deg(P) > \deg(Q)$, then use long division first, then Case 1, Case 2 or Case 3.

Example 2 (Similar to HW)

$$\int \frac{dx}{x^2+8x+19} = \int \frac{dx}{(x+4)^2+3} = \boxed{\begin{aligned} x+4 &= z(x) \\ dx &= dz \end{aligned}}$$

$$= \int \frac{dz}{z^2+3} = \int \frac{dz}{3\left(\frac{z^2}{3}+1\right)} = \frac{1}{3}\int \frac{dz}{\left(\frac{z}{\sqrt{3}}\right)^2+1} =$$

$$= \boxed{\begin{aligned} \frac{z}{\sqrt{3}} &= u \\ dz &= \sqrt{3}\,du \end{aligned}} = \frac{1}{3}\int \frac{\sqrt{3}\,du}{u^2+1} =$$

$$= \frac{\sqrt{3}}{3}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{3}}\arctan(u) + C =$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{z}{\sqrt{3}}\right) + C =$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x+4}{\sqrt{3}}\right) + C.$$

Example 3

Given
$$\int_0^5 \frac{3x-5}{x^2-5x+6}\,dx \quad -?$$

$$Q(x) = x^2-5x+6$$

$$D = 25-4\cdot 6 = 1 > 0$$

$$x_1 = \frac{5+1}{2} = 3, \quad x_2 = \frac{5-1}{2} = 2$$

The denominator $Q(x)$ factors as $(x-2)(x-3)$

$$\int_0^5 \frac{3x-5}{x^2-5x+6}\,dx = \int_0^5 \frac{3x-5}{(x-2)(x-3)}\,dx$$

$$\frac{3x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}; \quad A, B -?$$

$$3x-5 = A(x-3) + B(x-2) = Ax - 3A + Bx - 2B$$

$$3x-5 = (A+B)x - (3A+2B)$$

$$\begin{cases} A+B = 3 \\ -(3A+2B) = -5 \end{cases} \Rightarrow \begin{cases} A = 3-B \\ 3A+2B = 5 \end{cases}$$

$$3(3-B)+2B = 5$$

$$9-3B+2B = 5$$

$$\boxed{B = 4}, \quad \boxed{A = -1}$$

Thus, $\displaystyle\int_0^5 \frac{3x-5}{(x-2)(x-3)}\,dx = \int_0^5 \frac{-dx}{x-2} + \int_0^5 \frac{4\,dx}{x-3} =$

$$= -\int_0^5 \frac{dx}{x-2} + 4\int_0^5 \frac{dx}{x-3} = -\ln|x-2|\Big|_0^5 + 4\ln|x-3|\Big|_0^5 =$$

$$= -\ln 3 + \ln 2 + 4\ln 2 - 4\ln 3 = \ln\frac{2}{3} + 4\ln\frac{2}{3} = \ln\left[\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^4\right]$$

$\boxed{\text{Example 4}}$ $$\int \frac{(2x+3)}{x^2-2x+1}\,dx = \int \frac{(2x+3)}{(x-1)^2}\,dx$$

$$f(x) = \frac{P(x)}{Q(x)} = \frac{2x+3}{(x-1)^2} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$2x+3 = A_1(x-1) + A_2 = A_1 \cdot x + (A_2 - A_1)$$

$$\begin{cases} A_1 = 2 \\ A_2 - A_1 = 3 \end{cases} \qquad A_2 = A_1 + 3 = 5$$

$$\frac{2x+3}{(x-1)^2} = \frac{\cancel{2}^{\,x-1}}{x-1} + \frac{5}{(x-1)^2} = \frac{2x-2+5}{(x-1)^2} = \frac{2x+3}{(x-1)^2}$$

$$\int \frac{2x+3}{x^2-2x+1}\,dx = \int \frac{2\,dx}{x-1} + \int \frac{5\,dx}{(x-1)^2} =$$

$$= 2\ln|x-1| - \frac{5}{x-1} + C.$$

Check the answer:

Guess-and-Check Method:

$$\left(2\ln|x-1|\right)' = \begin{cases} [2\ln(x-1)]', & x-1 \geq 0 \\ [2\ln(1-x)]', & x-1 < 0 \end{cases} =$$

$$= \begin{cases} \frac{2}{x-1}, & x-1 \geq 0. \\ \frac{2}{1-x}\cdot(-1), & x < 1 \end{cases} = \begin{cases} \frac{2}{x-1}, & x \geq 1 \\ \frac{2}{x-1}, & x < 1 \end{cases}$$

$$\left(-\frac{5}{x-1}\right)' = -5\cdot\left(\frac{1}{x-1}\right)' = -5\cdot\left[\frac{1'\cdot(x-1) - 1\cdot(x-1)'}{(x-1)^2}\right] =$$

$$= -5\left[\frac{-1}{(x-1)^2}\right] = \frac{5}{(x-1)^2}.$$

Evaluate $\boxed{\text{Example 5}}$

$$\int \frac{x^3-x-2}{x^2-4}\,dx$$

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^3-x-2}{x^2-4}$$

$\deg(P) > \deg(Q) \Rightarrow$ use long division to get case 1, or case 2, or case 3.

$$\begin{array}{r}
x \\
x^2-4 \,\overline{)\, x^3 - x - 2} \\
-\,(x^3 - 4x) \\
\hline
3x - 2
\end{array}$$

$$x^3-x-2 = (x^2-4)\cdot x + (3x-2)$$

proper fraction

$$\frac{x^3-x-2}{x^2-4} = \frac{(x^2-4)\cdot x + (3x-2)}{x^2-4} = x + \frac{3x-2}{x^2-4} =$$

$$= x + \frac{3x-2}{(x-2)(x+2)}$$

degree of the numerator is less than degree of the denominator

$$\int \frac{x^3-x-2}{x^2-4}\,dx = \int \left(x + \frac{3x-2}{(x-2)(x+2)}\right)dx$$

$$\frac{3x-2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad \Big|\cdot(x-2)(x+2)$$

$$(3x-2) = A(x+2) + B(x-2) = Ax + 2A + Bx - 2B$$

$$(3x-2) = (A+B)\,x + 2(A-B)$$

$$\begin{cases} A+B = 3 \\ 2(A-B) = -2 \end{cases}$$

$A = 3 - B$

$A - B = -1$

$A = B - 1$

$3 - B = B - 1$

$2B = 4 \Rightarrow \boxed{B = 2}$ / $\boxed{A = 3 - 2 = 1}$

Thus, $\dfrac{3x-2}{(x-2)(x+2)} = \dfrac{1}{x-2} + \dfrac{2}{x+2}$

$$\int \left(x + \frac{3x-2}{(x-2)(x+2)}\right)dx = \int \left(x + \frac{1}{x-2} + \frac{2}{x+2}\right)dx =$$

$$= \int x\,dx + \int \frac{dx}{x-2} + 2\int \frac{dx}{x+2} = \frac{x^2}{2} + \ln|x-2| +$$

$$+ 2\ln|x+2| + C.$$

Example 6

$\int \frac{x^3+x^2+5}{x^2+4}$, Need to use long division, since the degree of the numerator is greater than the degree of the denominator

$$
\begin{array}{r l}
 & x+1 \quad \text{(quotient)} \\
\text{divisor} \rightarrow x^2+4 \,\big) & x^3+x^2+5 \\
 & -\;x^3+4x \\
\hline
 & x^2-4x+5 \\
 & -\;x^2 \quad\; +4 \\
\hline
 & -4x+1 \quad \text{(remainder)}
\end{array}
$$

Thus, $x^3+x^2+5 = (x^2+4)(x+1) + (1-4x)$

$$\frac{x^3+x^2+5}{x^2+4} = \frac{(x^2+4)(x+1)}{x^2+4} + \frac{1-4x}{x^2+4} = (x+1) + \frac{1-4x}{x^2+4}$$

$$\int \frac{x^3+x^2+5}{x^2+4}\,dx = \int \left(x+1 + \frac{1-4x}{x^2+4}\right) dx =$$

$$= \int x\,dx + \int dx + \int \frac{dx}{x^2+4} - 4\int \frac{x\,dx}{x^2+4} =$$

$$= \frac{x^2}{2} + x + I_2 - 4I_1 + C.$$

from Example 1
(up to multiplicative constant)

$$= \frac{x^2}{2} + x + \frac{1}{2}\arctan\frac{x}{2} - 4\cdot\left(\frac{1}{2}\ln|x^2+4|\right) + C =$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\arctan\left(\frac{x}{2}\right) - 2\ln|x^2+4| + C.$$

$$f(x) = \frac{P(x)}{Q(x)}$$

Recall: We assumed that $\deg(P) < \deg(Q)$ and $\deg(Q) \leq 2$.

- What if $\deg(P) = \deg(Q)$?

$\deg(Q) = 1$ obtain the same expressions as in $Q(x)$

$\boxed{\text{Example 1}}$ $$\int \frac{x\,dx}{x+2} = \int \frac{x+2-2}{x+2}\,dx =$$

$$= \int dx - 2\int \frac{dx}{x+2} = x - 2\ln|x+2| + C.$$

Partial fra.

Example: $\deg(P) = \deg(Q) = 2$

Case 1

$$\int \frac{x^2+9}{x^2-9}\,dx = \int \frac{x^2-9+18}{x^2-9}\,dx = \int dx + 18\int \frac{dx}{x^2-9} \circledast$$

$$\int \frac{dx}{x^2-9} = \int \frac{A\,dx}{x-3} + \int \frac{B\,dx}{x+3}, \text{ since}$$

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3} =$$

$$= \frac{1}{6(x-3)} - \frac{1}{6(x+3)} = \cancel{x+3-x+3}$$

$$A\cancel{x} + 3A + Bx - 3B = 1$$

$$A + B = 0 \quad A = -B$$

$$3A - 3B = 1$$

$$-6B = 1 \quad B = -\frac{1}{6}$$

$$A = \frac{1}{6}.$$

Thus, $$\int \frac{dx}{x^2-9} = \frac{1}{6}\int \frac{dx}{x-3} - \frac{1}{6}\int \frac{dx}{x+3} =$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C.$$

$$\circledast\ x + \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

$$\int \frac{u\,du}{u^2+a^2} = \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln|z| + c =$$

$$u^2+a^2 = z$$

$$2u\,du = dz$$

$$u\,du = \frac{dz}{2}.$$

$$= \frac{1}{2}\ln|u^2+a^2| + c.$$

$$\int \frac{du}{u^2+a^2} = \int \frac{du}{a^2\left(\frac{u^2}{a^2}+1\right)} =$$

$$= \frac{1}{a^2}\int \frac{du}{\left(\frac{u}{a}\right)^2+1}$$

$$\frac{u}{a} = z$$

$$du = a\,dz$$

$$= \frac{1}{a^2}\int \frac{a\,dz}{z^2+1} = \frac{1}{a}\int \frac{dz}{z^2+1} =$$

$$= \frac{1}{a}\arctan z + c = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$$